Synthesis of Nonlinear Stiffness Profile with a Varying Radius Cable Spool

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Abstract— In this paper, we present a cable mechanism which synthesizes a nonlinear rotational spring from a linear spring. The nonlinearity is realized by winding the cable around a spool which has a varying radius. We show that for a given nonlinear torque profile (synthesis objective) there is an explicit geometric expression of the shape of the spool which synthesizes this profile. We present the geometry of the problem, explain the method to calculate the shape of the spool, and show an example of synthesis.

Key Words: nonlinear spring, cable-spool system, profile synthesis

1. Introduction

The motion of a robotic system emerges through the interaction of the control system, the mechanical system, and the environment. Although mechanical synthesis and control theory where traditionally two distinct fields of study, recent research in robotic engineering aim at considering both the control system and the mechanical system as design parameters to generate the motion [1, 2, 3]. In this paradigm, the robot's motion should result as much as possible from the natural dynamics of the mechanism, in order to realize smooth, natural and low-energy consuming movements.

One of the key design parameters of such systems is the stiffness of the joint angles. On one hand, high stiffness is required to achieve an accurate and high speed motion of the robot. On the other hand, softness is required to ensure a smooth and safe interaction with the environment. Therefore, we need to synthesize a mechanism with nonlinear stiffness profiles. This can be achieved either by finite-Degree Of Freedom (DOF) methods, which consist in optimizing a finite number of parameters of the mechanism [4], or infinite-DOF methods, which perform a continuous calculation of the mechanism. Examples of infinite-DOF methods are cam mechanisms, rolamite springs [5], and mechanism with bearings rolling of a curved frame [6].

In this paper, we present a method to synthesize a nonlinear rotational spring from a linear spring using a cable mechanism. The cable, connected to the linear spring, is wound around a non-circular spool as shown of Fig. 1. When the spool rotates, the distance from the spool axis to the tangency point of the cable is modified, thus modifying the speed at which the spring is displaced. This results in a nonlinear relationship between the angle of the spool and the torque applied by the spring. Depending on the shape of the spool, various stiffness profiles can be realized. This paper is organized as follows: Section 2. presents the mechanism, the notations and defines the synthesis objective. Section 3. explains the calculation method of the spool. Section 4. shows an example of synthesis and explains the method used to verify the torque function realized by the mechanism.

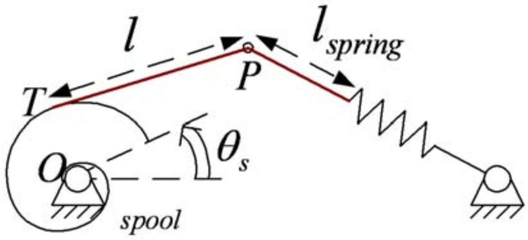

Fig.1 Mechanism with a varying radius cable spool

2. Varying radius cable spool

We consider the mechanism shown in Fig. 1. A linear spring is connected to a cable which goes trough a pulley P and is wound around a non-circular spool. We note O the axis of the spool, θ_s the angular position of the spool with respect to the reference frame and q the displacement of the linear spring with respect to its natural length. Because the spool is not circular, the torque τ created in O by the tension of the cable is a nonlinear function of θ_s. In this paper, we propose a method to compute the shape of the spool which synthesizes a given nonlinear torque profile $\tau(\theta_s)$.

Fig. 2 shows the details of the spool. We note P the pulley and T the point where the cable is tangent to the spool. We note l_{spool} the length of cable wound around the spool, l the length TP and l_{spring} the length of cable from the pulley to the spring. The total length of cable in the mechanism is noted L. By definition, $L = l_{spool} + l + l_{spring}$. We note $r = OT$ the varying radius of the spool and R the distance OP. In the coordinate frame attached to the spool, we note θ_p the angular position of the pulley and θ_r the angular position of the varying radius r. Finally, we note α the angle of TP with respect to the perpendicular to OT.

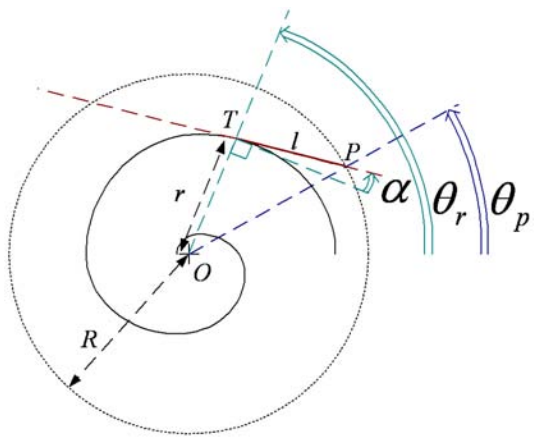

Fig.2 Details of the spool

As mentioned before, we want to calculate the shape of the spool which synthesizes a given torque profile $\tau(\theta_s)$. Assuming that $\tau(\theta_s) > 0$, the Principe of Virtual Work gives the following relationship:

$$-\tau(\theta_s)\, d\theta_s = -k\, q\, dq \tag{1}$$

where k is the spring constant of the linear spring. Since this system has only one DOF, we can derive the following relationship from Eq. (1):

$$\frac{dq}{d\theta_s} = \frac{\tau(\theta_s)}{\sqrt{2\, k \int_0^{\theta_s} \tau(u)\, du + q_0^2}} \tag{2}$$

where q_0^2 is the value of q when $\theta_s = 0$. $\frac{dq}{d\theta_s}$ can be written as:

$$\frac{dq}{d\theta_s} = \frac{dq}{dl_{spring}} \frac{dl_{spring}}{d\theta_p} \frac{d\theta_p}{d\theta_s} \tag{3}$$

l_{spring} and θ_s satisfy the following relationships:

$$l_{spring} + q = l_{spring,0} \tag{4}$$
$$\theta_s + \theta_p = \theta_{p,0} \tag{5}$$

where $l_{spring,0}$ is the length of cable between the pulley and the spring when it is at its natural length and $\theta_{p,0}$ is the angular position of the pulley when $\theta_s = 0$. From Eqs. (3), (4) and (5) we obtain the following equation:

$$\frac{dq}{d\theta_s} = \frac{dl_{spring}}{d\theta_p} \tag{6}$$

From now, we use the notation $\frac{dl_{spring}}{d\theta_p} = J(\theta_p)$. Using Eqs. (2) and (6), we can calculate $J(\theta_p)$ as:

$$J(\theta_p) = \frac{\tau(\theta_{p,0} - \theta_p)}{\sqrt{2\, k \int_0^{(\theta_{p,0}-\theta_p)} \tau(u)\, du + q_0^2}} \tag{7}$$

$J(\theta_p)$ defines the input/output relationship that the spool must achieve to synthesize the torque profile $\tau(\theta_s)$.

3. Calculus method

Considering that the kinematic input of the spool is θ_p, the output l_{spring}, and the objective input/output relationship $J(\theta_p)$, we need to determine the relationships between θ_p and r and between θ_p and θ_r to calculate the shape of the spool. The spool boundary is constrained by two geometric conditions: the first one is the tangency of the cable to the spool in T, the second one is the conservation of the total length of cable in the system L.

The tangency condition of the cable in T imposes the following condition:

$$\frac{dr}{d\theta_r} = -r\, \tan\alpha \tag{8}$$

For later use, we give the expression of α.

$$\cos\alpha = \frac{R}{l}\, \sin(\theta_r - \theta_p) \tag{9}$$
$$\sin\alpha = \frac{R}{l}\, \cos(\theta_r - \theta_p) - \frac{r}{l} \tag{10}$$
$$\tan\alpha = \frac{R\, \cos(\theta_r - \theta_p) - r}{R\, \sin(\theta_r - \theta_p)} \tag{11}$$

We now write the conservation of the total length L of the cable .

$$\frac{dL}{d\theta_p} = \frac{dl_{spool}}{d\theta_p} + \frac{dl}{d\theta_p} + \frac{dl_{spring}}{d\theta_p} = 0 \tag{12}$$

We detail the calculation of the three terms of this equation.

First term:

We write $\frac{dl_{spool}}{d\theta_p}$ as:

$$\frac{dl_{spool}}{d\theta_p} = \frac{dl_{spool}}{d\theta_r} \frac{d\theta_r}{d\theta_p} \tag{13}$$

We consider the displacement of the tangency point T for a small variation of θ_r, as shown in Fig. 3. From the law of cosines, the length δl is given by:

$$\delta l^2 = (r(\theta_r))^2 + (r(\theta_r + \delta\theta_r))^2 - 2r(\theta_r) r(\theta_r + \delta\theta_r) \cos(\delta\theta_r) \tag{14}$$

We calculate the second order Taylor expansion of this expression. After simplifications, we obtain:

$$\delta l^2 = \left(\left(\frac{dr}{d\theta_r} \right)^2 + r^2 \right) \delta\theta_r^2 + \mathcal{O}\left((\delta\theta_r)^3 \right) \tag{15}$$

By substituting (8) in (15), we obtain:

$$\delta l^2 = \frac{r^2}{\cos^2(\alpha)}\, \delta\theta_r^2 + \mathcal{O}\left(\delta\theta_r^3\right) \tag{16}$$

From equation (16), $\frac{dl_{spool}}{d\theta_r}$ is given by:

$$\frac{dl_{spool}}{d\theta_r} = -\lim_{\delta\theta_r \to 0} \frac{\delta l}{\delta\theta_r} = -\frac{r}{\cos\alpha} \tag{17}$$

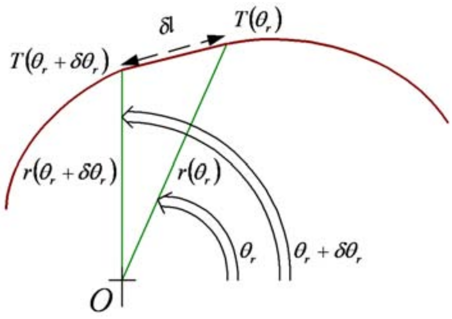

Fig.3 Small variation of l_{spool}

Finaly, the expression of $\frac{dl_{spool}}{d\theta_p}$ is

$$\frac{dl_{spool}}{d\theta_p} = -\frac{r}{\cos\alpha}\frac{d\theta_r}{d\theta_p} \quad (18)$$

Second term:

From the law of cosines applied to the triangle POT, the expression of l is:

$$l = \sqrt{R^2 + r^2 - 2\,R\,r\,\cos(\theta_r - \theta_p)} \quad (19)$$

The partial derivatives of l with respect to r, θ_r and θ_p are:

$$\frac{\partial l}{\partial \theta_r} = \frac{R\,r\,\sin(\theta_r - \theta_p)}{l} \quad (20)$$

$$\frac{\partial l}{\partial r} = \frac{r - R\,\cos(\theta_r - \theta_p)}{l} \quad (21)$$

$$\frac{\partial l}{\partial \theta_p} = -\frac{R\,r\,\sin(\theta_r - \theta_p)}{l} \quad (22)$$

The total derivative of l with respect to θ_p is:

$$\frac{dl}{d\theta_p} = \frac{\partial l}{\partial \theta_r}\frac{d\theta_r}{d\theta_p} + \frac{\partial l}{\partial r}\frac{dr}{d\theta_p} + \frac{\partial l}{\partial \theta_p} \quad (23)$$

Using Eqs. (9), (11), (19), (20), (21), (22), (23) and (8), we obtain:

$$\frac{dl}{d\theta_p} = \frac{r}{\cos(\alpha)}\frac{d\theta_r}{d\theta_p} - \frac{R\,r}{l}\sin(\theta_r - \theta_p) \quad (24)$$

Third term:

As explained in the previous section, we defined $J(\theta_p)$ (which can be calculated by Eq. (7)) as:

$$\frac{dl_{spring}}{d\theta_p} = J(\theta_p) \quad (25)$$

By substituting Eqs. (18), (24) and (25) in (12), we obtain the following equation:

$$\frac{R\,r}{l}\sin(\theta_r - \theta_p) = J(\theta_p) \quad (26)$$

This is a first geometric equation linking r, θ_r and θ_p.

By taking the square of this equation, calculating the total derivative with respect to θ_p and combining with Eq. (8), we obtain a second geometric equation. Because of space limitation, we spare the details of the calculus.

$$\frac{R^2\,r^2}{l^4}\sin(\theta_r - \theta_p) * \left[\cos(\theta_r - \theta_p)\,(R^2 + r^2) - (1 + \cos^2(\theta_r - \theta_p))\,R\,r\right] + J(\theta_p)\,J'(\theta_p) = 0 \quad (27)$$

where $J'(\theta_p) = \frac{dJ(\theta_p)}{d\theta_p}$

For a given set of θ_p, the spool boundary is defined by the set of (r, θ_r) solutions of the system $\{(26), (27)\}$. We now solve this system to find the explicit solution. First, we solve Eq. (26) and write θ_r as a function of θ_p and r.

$$\theta_r = \theta_p + \arccos\left(\frac{J^2(\theta_p)}{R\,r}\left(1 \pm \sqrt{\left(1 - \left(\frac{r}{J(\theta_p)}\right)^2\right)\left(1 - \left(\frac{R}{J(\theta_p)}\right)^2\right)}\right)\right) \quad (28)$$

The $\pm$ sign in the above equation comes from the resolution of a second order equation (which gives two solutions). Using Eqs. (10) and (28) we can prove that the quantity below the square root is null if and only if $\sin\alpha = 0$. Thus, when calculating the spool boundary, the symbol $\pm$ has to be changed from $+$ to $-$ (or the opposite) each time that $\alpha = 0$ occurs to ensure that $\frac{d\alpha}{d\theta_p}$ is continuous. We now substitute this expression in Eq. (27) and solve the equation to obtain the explicit expression of r with respect to θ_p.

$$r(\theta_p) = \sqrt{J^2(\theta_p) + \frac{J'^2(\theta_p)\,(R^2 - J^2(\theta_p))}{\left(J'(\theta_p) \mp \sqrt{R^2 - J^2(\theta_p)}\right)^2}} \quad (29)$$

Eqs. (28) and (29) define the explicit solution of the spool boundary for a given function $J(\theta_p)$. Note that for a single function $J(\theta_p)$ the system has two solutions, distinguished by the $\mp$ sign in Eq. (29). The reason to the existence of two distinct solutions is that the spool has always two tangents passing through the pulley.

An important remark is that a necessary condition so that the system $\{(26),(27)\}$ has a solution is $J(\theta_p) \leq R$. Under this condition, the radius of the spool verifies the relationship $J(\theta_p) \leq r(\theta_p) \leq R$. Furthermore, since the tension in the cable must stay positive, the objective torque profile $\tau(\theta_s)$ must be strictly positive.

4. Verification of calculus method

Using equations (29) and (28), we compute the shape of the spool realizing a nonlinear torque profile. Fig. 4 shows the shape of the spool. The red circle, which diameter is R, shows the outer limit of the domain inside of which r can vary. Fig. 5 shows the value of the radius r with respect to θ_p.

To verify the accuracy of the synthesis, we use a numeric method to find the tangency point T for each

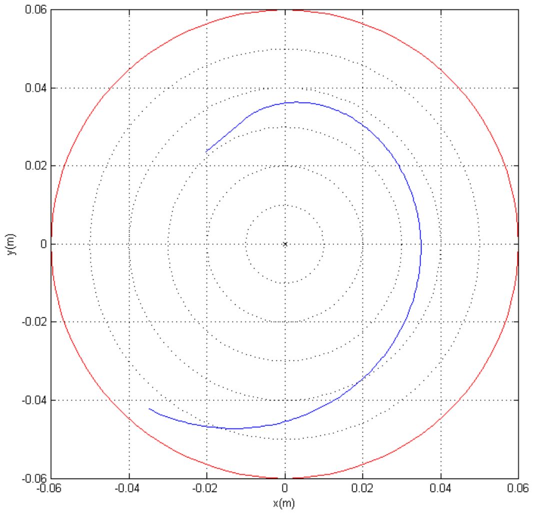

Fig.4 Spool synthesizing the torque profile

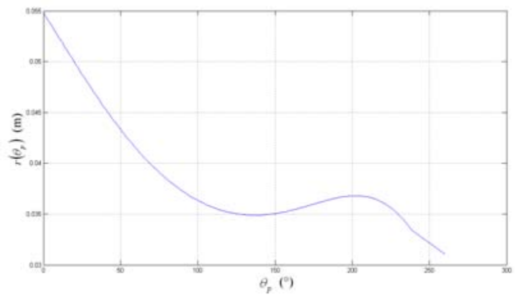

Fig.5 Varying radius r with respect to θ_p

angular position of the spool θ_s. From the coordinates of T, we obtain the values of r and θ_r, then calculate the values of l and α. Without loss of generality, we assume that cable is attached to the spool at the angular position $\theta_{r,max}$ (greatest element of $(\theta_{r,i})_i$). We calculate l_{spool} by summing the lengths of the segments of the spool boundary from θ_r to $\theta_{r,max}$. By subtracting l and l_{spool} from L (which is constant), we get the value of l_{spring}. From l_{spring}, we know the displacement of the linear spring and thus the tension of the cable. With the angle α calculated before, we compute the torque created by the cable on the spool axis.

The comparison of the theoretical torque (solid line) and synthesized torque (x-mark line) is shown on Fig. 6. Fig. 6 shows that the torque profile computed numerically fits the theoretical torque profile. These results show that from a given torque profile $\tau(\theta_s)$, we can calculate the shape of the cable spool which synthesizes this torque profile.

5. Conclusion

In this paper, we proposed a cable mechanism which uses a varying radius spool to synthesize a nonlinear rotational spring from a linear spring. We explained the methodology to compute the shape of the spool

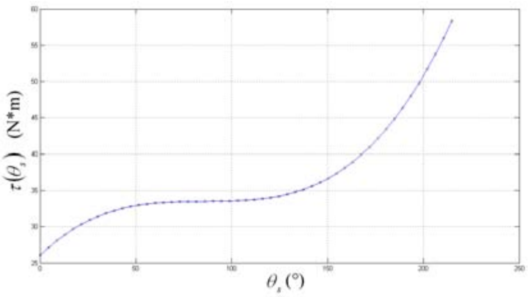

Fig.6 Torque with respect to θ_s

and showed that the synthesis problem has an explicit geometric solution, that is, for a given torque profile $\tau(\theta_s)$, there is an explicit expression of the shape of the spool which realizes this function. We presented an exemple of synthesis of a nonlinear torque profile.

Future work will include experiments of the cable system and further theoretical study.

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