Joint Design of Model-Subspace based State-Space Identification and Control

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ABSTRACT: Recently, the importance of the joint design of identification and control has been recognized, and several controller design methods based on the iteration of identification and controller re-design have been proposed. In these methods, the frequency weighted identification plays an important role. On the other hand, as a powerful identification method, subspace state-space system identification (4SID) method has been proposed. However, it is difficult to use the frequency weight in conventional 4SID methods. In this paper, we propose a new frequency weighted subspace state-space system identification method for the joint design of identification and control. First, we give a new 4SID method using not only the input-output data of the system but also the nominal model data. Second, we show how to introduce the frequency weight to our identification method, which is relevant to the cost function for control. And third, we evaluate the effectiveness of the proposed method by a numerical example using an inverted pendulum system.

Key Words: system identification, subspace method, closed-loop identification, joint design of identification and control

1 Introduction

Recently, much attention is paid to the joint design of identification and control [1]~[6]. In this method, we iterate the closed-loop identification and controller design for the purpose of obtaining the high performance of the closed loop system, and the frequency weighted identification plays an important role.

On the other hand, the schemes which identify state-space models directly from the input-output data are known as ***subspace state-space system identification*** (4SID) method and have received much attention[7]~[13]. One of the merits of 4SID is that it is applicable to MIMO systems as well as SISO systems. Furthermore, these methods use reliable numerical computation, and the computational complexity of the algorithms is lower than classical prediction error methods (PEM). The optimality of 4SID methods is known as the quadratic optimization of mult-steps ahead prediction errors [14]. However, it is difficult to use the frequency weight in the framework of the conventional 4SID. Moreover, the 4SID methods have still some other problems in practical situations. For instance, they are more sensitive to noise than PEM, and require the property that the noise vectors are white, so it is difficult to identify systems operating in closed-loop exactly with existing 4SID methods. Because of the above demerits, we can not use the 4SID as the powerful tool of the joint design of identification and control.

So in this paper, we propose a new subspace based identification method which is related to the control relevant cost function and can attenuate noises by using the nominal model data as the instrumental variable. In addition, since the nominal model data are purely uncorrelated with noise in any cases, this method can be applied to the closed-loop identification problem with the similar formulation to the open-loop one. Furthermore, by introducing the weighting function to this method, we propose a joint design method of subspace system identification and control. Finally, the effectiveness of our method is evaluated by numerical examples.

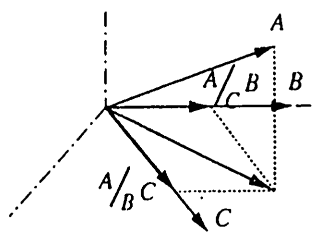

Figure 1: Projections

In this paper, we use the following notations. A/B is the projection of the matrix A into the row space of B, so that

$$A/B = AB^T(BB^T)^\dagger B. \tag{1}$$

$A/_C B$ is the projection of A along the row space of B into the row space of C[10], so that

$$A/_C B = [A/C^\perp]\cdot[B/C^\perp]^\dagger B, \tag{2}$$

where $A/B^\perp$ is the projection of the A into the orthogonal complement to Image(B)(see Fig.1). $[\cdot]^{\dagger}$ and $\|\cdot\|$ denotes the Moore-Penrose pseudo-inverse and the Frobenius norm respectively. $A(i:j,k:\ell)$ denotes the submatrix of A consisting of rows i to j and columns k to ℓ, and also $A(i:j,:)$ denotes the submatrix of rows i to j.

2 Problem formulation

2.1 Joint design problem

Consider the linear time-invariant system P given by the following state-space formula.

$$P:\left\{\begin{array}{lcl} x_p[k+1] & = & A_p x_p[k] + B_p u_p[k] + w[k] \\ y_p[k] & = & C_p x_p[k] + D_p u_p[k] + v[k] \end{array}\right. \tag{3}$$

where $x_p \in R^n$(the state vector), $u_p \in R^m$(the input vector) and $y_p \in R^\ell$(the output vector). $v \in R^\ell$ and $w \in R^n$ are assumed to be zero mean noise vectors.

Our goal is to design the controller $K_i, (i = 1, 2, \cdots)$ which minimizes $J_c(P, K_i)$ as

$$J_c(P,K_i) := \left\| \begin{array}{c} W_s\Phi(P,K_i) \\ W_t\Psi(P,K_i) \end{array} \right\|_2 \tag{4}$$

$$\Phi(P,K_i) = P(I+K_iP)^{-1} \tag{5}$$

$$\Psi(P,K_i) = K_iP(I+K_iP)^{-1} \tag{6}$$

by iterating the closed-loop identification and controller design. Here W_s and W_t are frequency depended weighting functions which are given a priori. First, we make the following assumptions.

[Assumption 1] The nominal model P_{m1} of P

$$P_{m1}:\left\{\begin{array}{lcl} x_{m1}[k+1] & = & A_{m1} x_{m1}[k] + B_{m1} u_{m1}[k] \\ y_{m1}[k] & = & C_{m1} x_{m1}[k] + D_{m1} u_{m1}[k] \end{array}\right. \tag{7}$$

is given.

[Assumption 2] The controller K_1 designed by

$$K_1 = \arg\min_K J_c(P_{m1}, K) \tag{8}$$

can stabilize P.

[Assumption 3] The initial state of the plant is zero.

2.2 Controller design problem

The controller K_i is designed based on (4). However, because P is unknown, it is obtained by

$$K_i = \arg\min_K J_c(P_{mi}, K) \tag{9}$$

alternatively. Here P_{mi} is the i-th model identified in the i-th iteration stage.

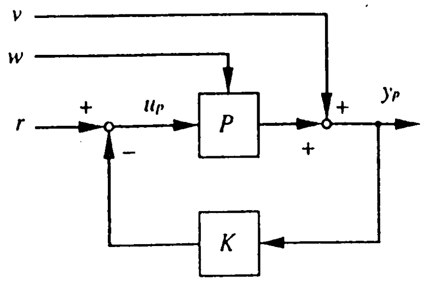

Figure 2: Closed loop system

2.3 Closed-loop identification problem

As for the criterion function in (4) in the i-th iteration stage, the following inequality is satisfied.

$$J_c(P,K_i) \le \left\| \begin{array}{c} W_s\Phi(P_{mi},K_i) \\ W_t\Psi(P_{mi},K_i) \end{array} \right\|_2 + \left\| \begin{array}{c} W_s(\Phi(P,K_i)-\Phi(P_{mi},K_i)) \\ W_t(\Psi(P,K_i)-\Psi(P_{mi},K_i)) \end{array} \right\|_2 \tag{10}$$

Because the first term of the right hand side is minimized by K_i in (9), we aim at minimizing the second term by the closed-loop identification. It is the basic idea of the joint design of identification and control[1]. Moreover J_c consists of two transfer functions Φ and Ψ. Because Φ corresponds to the performance of the closed loop system such as the disturbance rejection, while Ψ relates to the stability for the model uncertainty (which is difficult to idetify), we adopt the criterion function for the identification to be

$$P_{m(i+1)} = \arg\min_{P_m} J_{id}(P_m, K_i) \tag{11}$$

$$J_{id} = \|W_s(\Phi(P,K_i) - \Phi(P_m,K_i))\|_2 \tag{12}$$

3 Identification method

As mentioned above, in the joint design of identification and control, we must identify the model P_m using frequency weighting function. However in the conventional 4SID methods, we can not introduce the weighting function. So in this section, we propose a new 4SID method of which cost function is related to the H_2 norm, and introduce the frequency weighted identification.

3.1 Model-subspace based state space identification

Consider the closed loop system shown in Fig.2 where r is a reference signal for identification and assumed to be uncorrelated with w and v. Then the following input-output equation is obtained.

$$Y_p = \Gamma_p X_p + H_p U_p + FW + V \tag{13}$$

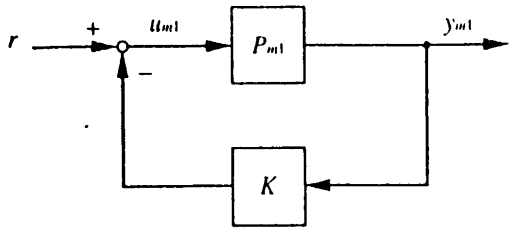

Figure 3: Closed loop system using the model

where we define

$$Y_p := \begin{bmatrix} y_p[1] & y_p[2] & \cdots & y_p[N] \\ y_p[2] & y_p[3] & \cdots & y_p[N+1] \\ \vdots & \vdots & & \vdots \\ y_p[j] & y_p[j+1] & \cdots & y_p[N+j-1] \end{bmatrix} \quad (14)$$

$$X_p := \begin{bmatrix} x_p[1] & x_p[2] & \cdots & x_p[N] \end{bmatrix} \quad (15)$$

$$\Gamma_p := \begin{bmatrix} C_p \\ C_pA_p \\ \vdots \\ C_pA_p^{j-1} \end{bmatrix} \quad (16)$$

$$H_p := \begin{bmatrix} D_p & & 0 \\ C_pB_p & D_p & \\ \vdots & \ddots & \\ C_pA_p^{j-2}B_p & \cdots \cdots & D_p \end{bmatrix} \quad (17)$$

$$F := \begin{bmatrix} 0 & & 0 \\ C_p & 0 & \\ \vdots & \ddots & \\ C_pA_p^{j-2} & C_pA_p^{j-3} \cdots & 0 \end{bmatrix} \quad (18)$$

and U_p, W and V are defined by u_p, w and v respectively in the same way as Y_p. The equation (13) can be split into the deterministic part $[\cdot]^d$ effected by r and the stochastic part $[\cdot]^s$ influenced by the noises w and v as follows.

$$Y_p = Y_p^d + Y_p^s \quad (19)$$
$$Y_p^d = \Gamma_p X_p^d + H_p U_p^d \quad (20)$$
$$Y_p^s = \Gamma_p X_p^s + H_p U_p^s + FW + V \quad (21)$$

In the same way, the following model input-output equation is obtained from the closed loop system shown in Fig.3.

$$Y_{m1} = \Gamma_{m1} X_{m1} + H_{m1} U_{m1} \quad (22)$$

Here, we consider the projection of Y_p into the row space of $[X_{m1}^T \ U_{m1}^T]^T$. By the projection of (19), we obtain

$$\widehat{Y}_p = \Gamma \widehat{X}_p^d + H_p \widehat{U}_p^d + \widehat{Y}_p^s \quad (23)$$

$$\widehat{[\cdot]} := [\cdot] \Big/ \begin{bmatrix} X_m \\ U_m \end{bmatrix} \quad (24)$$

Since w and v are uncorrelated with r, we have

$$\lim_{N\to\infty} \frac{1}{N} \left\| \widehat{Y}_p^s \right\| = 0 \quad (25)$$

Therefore, we can rewrite (23) as follows.

$$\widehat{Y}_p = \Gamma \widehat{X}_p^d + H_p \widehat{U}_p^d \quad (26)$$

The noise attenuation is achieved in closed-loop identification, which is the effect of using instrumental variable.

Now, (26) can be decomposed into the following X_{m1} and $\widehat{U}_p^d$ components.

$$\widehat{Y}^p /_{\widehat{U}_p^d} X_{m1} = \Gamma_p \widehat{X}_p^d /_{\widehat{U}_p^d} X_m =: LX_{m1} \quad (^\exists L) \quad (27)$$

$$\widehat{Y}^p /_{X_{m1}} \widehat{U}_p^d = \Gamma_p \widehat{X}_p^d /_{X_{m1}} \widehat{U}_p^d + H_p \widehat{U}_p^d =: M\widehat{U}_p^d \quad (^\exists M) \quad (28)$$

Then if P_{m1} has same order as P has, we can consider L as the extended observability matrix of the system P with the state Tx_p^d such that

$$T := X_{m1} \left(\widehat{X}_p^d /_{\widehat{U}_p^d} X_{m1} \right)^\dagger \quad (29)$$

Let the coefficient matrices of the system P be

$$(\widehat{A}_p, \widehat{B}_p, \widehat{C}_p, \widehat{D}_p) = (T^{-1}A_pT, T^{-1}B_p, C_p, D_p) \quad (30)$$

then $\widehat{A}_p$ and $\widehat{C}_p$ can be computed from the following set of equation.

$$L(1:\ell(j-1))\widehat{A}_p = L(\ell+1:\ell j,:) \quad (31)$$

$$\widehat{C}_p \sum_{k=1}^{j} \widehat{A}_p^{k-1} = \sum_{k=1}^{j} L(\ell(k-1)+1:\ell k,:) \quad (32)$$

On the other hand, since multiplication of (28) from the left by $L^\perp$ yields

$$L^\perp M = L^\perp H_p \quad (33)$$

$\widehat{B}_p$ and $\widehat{D}_p$ can be calculated from this equation. The algorithm of the proposed method is shown in Appendix A.

3.2 The merit of the proposed method

In the conventional 4SID[9], we split the input-output equation (13) into the future part $[\cdot]^f$ and past part $[\cdot]^p$ as follows.

$$\begin{bmatrix} Y_p^p \\ Y_p^f \end{bmatrix} = \begin{bmatrix} \widetilde{\Gamma} \\ \widetilde{\Gamma} A_p^{\frac{i}{2}} \end{bmatrix} X_p^p + \begin{bmatrix} H_1 & 0 \\ H_2 & H_1 \end{bmatrix} \begin{bmatrix} U_p^p \\ U_p^f \end{bmatrix} \quad (34)$$

$$+ \begin{bmatrix} F_1 & 0 \\ F_2 & F_1 \end{bmatrix} \begin{bmatrix} W^p \\ W^f \end{bmatrix} + \begin{bmatrix} V^p \\ V^f \end{bmatrix} \quad (35)$$

And project Y_p^f into the row space of $[U_p^{pT} \ U_p^{fT} \ Y_p^{pT}]^T$. By this projection, the prediction $\widehat{Y}_p^f$ of Y_p^f is given as follows.

$$\widehat{Y}_p^f = \widehat{L} Y_p^p + \widehat{M}_1 U_p^p + \widehat{M}_2 U_p^f \quad (36)$$

$$(\widehat{L}_1, \widehat{M}_1, \widehat{M}_2) = \arg \min_{\widehat{L}_1, \widehat{M}_1, \widehat{M}_2} \left\| Y_p^f - \widehat{Y}_p^f \right\| \quad (37)$$

Then if w and v are uncorrelated with u_p and if both w and v are Gaussian distributed white noises,

$$\lim_{N\to\infty} \widehat{L} = \widetilde{\Gamma} \quad (38)$$

is satisfied and $\hat{Y}_p^f$ converges to the deterministic part of Y_p^f. However when w or v is correlated with u, or v or w is not white noise, both Y_p^f and Y_p^p are correlated with V^p or W^p. Therefore (38) is not satisfied and $\hat{Y}_p^f$ does not converge to the deterministic part of Y_p^f. Because in closed-loop identification, the noises have correlation with u_p and are not white, the accurate model is not obtained by the conventional 4SID in the closed-loop identification.

On the other hand, in the proposed method, the noise attenuation is achieved in spite of the closed-loop identification because of the folloing two reasones. One is that u_{m1} is uncorrelated with v and w, the other is that we use x_{m1} and u_{m1} as the instrumental variable.

3.3 Optimality of the proposed method

In this section, we use the following function.

[Assumption 4] The model P_{m1} is sufficiently close to the real plant P.

As mentioned above, the prediction $\hat{Y}_p$ of Y_p in (26) converges to the deterministic part of Y_p and the criterion function in (A.1) is equal to the minimization of

$$\left\|\hat{Y}_p - Y_p^d\right\| = \|Y_p^s\| \quad (39)$$

Because (39) is a quadratic norm, if r is the Gaussian distributed white signal, we can regard the cost function of the proposed method as the following H_2 norm criterion,

$$P_{m2} = \arg\min_{P_m} \|\Phi(P, K_i) - \Phi(P_m, K_i)\|_2 \quad (40)$$

which is the effect of using the nominal model data and attenuating the noise from not only the output data Y_p but also the input data U_p.

3.4 Introduction of the weighting function

In the PEM identification, when we use the weighting function W_{PEM} which satisfies

$$W_{PEM}P = PW_{PEM} \quad (41)$$

we get y_w and u_w such that

$$y_w = W_{PEM}y_p \ (= W_{PEM}Pu_p) \quad (42)$$
$$u_w = W_{PEM}u_p \quad (43)$$

and adopt them to PEM. By using y_w and u_w, we can obtain P_{m2} such that

$$P_{m2} = \arg\min_{P_m} \|W_{PEM}(P - P_m)\|_2 \quad (44)$$

In the same way, in the proposed method, we can introduce the weighting function W_{PR}, $\overline{W}_{PR}$ and $\widetilde{W}_{PR}$such that

$$W_{PR}\Phi(P, K_i) = \Phi(P, K_i)\overline{W}_{PR} \quad (45)$$
$$\widetilde{W}_{PR}(I + K_iP)^{-1} = (I + K_iP)^{-1}\overline{W}_{PR} \quad (46)$$

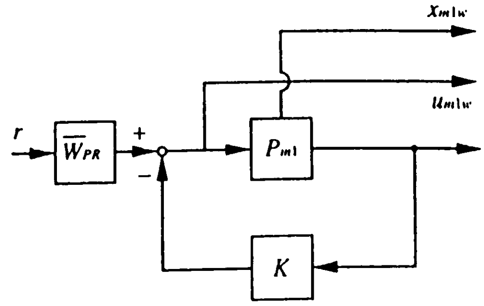

Figure 4: Weighted closed loop system using a model

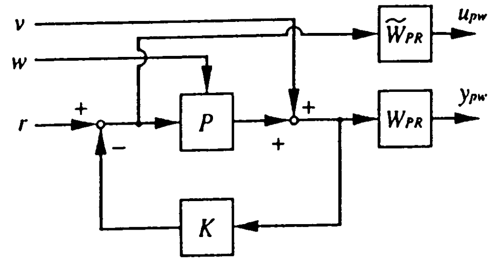

Figure 5: Weighted closed loop system

By using x_{m1w}, u_{m1w} obtained from the closed-loop simulation shown in Fig.4, y_{pw} and u_{pw} obtained from Fig.5 we can identify the model P_{m2} such that

$$P_{m2} = \arg\min_{P_m} \|W_{PR}(\Phi(P, K) - \Phi(P_m, K))\|_2 \quad (47)$$

which is the weighted model subspace based state space identification.

4 Joint design method

The algorithm of the joint design method is as follows

[Step 1] By using P_{m1}, we design the controller K_1 in (9).

[Step 2] In general case, $\overline{W}_s$ and $\widetilde{W}_s$ do not exist satisfying

$$W_s\Phi(P, K_1) = \Phi(P_{m2}, K)\overline{W}_s \quad (48)$$
$$\widetilde{W}_s(I + K_1P)^{-1} = (I + K_1P)^{-1}\overline{W}_s \quad (49)$$

So we consider the weighting function $\tilde{w}_s$ such that

$$\tilde{w}_s = \arg\min_{w} \|\tilde{w}_sI_\ell\Phi(P, K) - W_s\Phi(P, K)\|_2 \quad (50)$$

and set W_{PR}, $\overline{W}_{PR}$ and $\widetilde{W}_{PR}$ as

$$W_{PR} = \tilde{w}I_\ell, \quad \overline{W}_{PR} = \widetilde{W}_{PR} = \tilde{w}_sI_m \quad (51)$$

approximately. And get P_{m2} from (47) by using x_{m1w}, u_{m1w}, y_{pw} and u_{pw} obtained from Fig.4 and Fig.5. If P is a SISO system then

$$W_{PR} = \overline{W}_{PR} = \widetilde{W}_{PR} = W_s \quad (52)$$

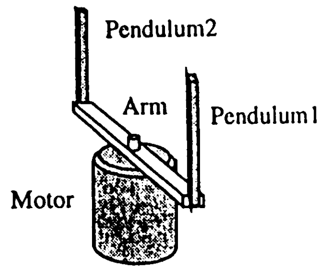

Figure 6: Parallel inverted pendulums system

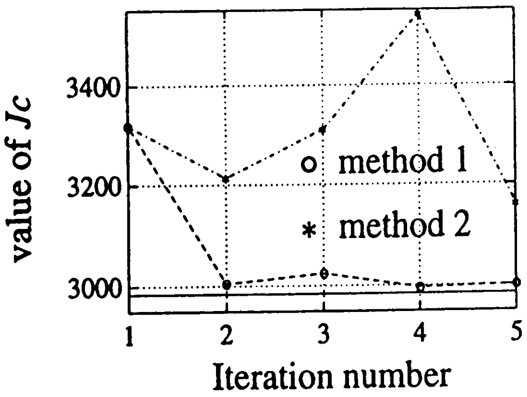

Figure 7: The value of J_c

[Step 3] Replace P_{m1} to P_{m2} and go to [Step 1] iteratively.

5 Numerical example

In this example, we show the effectiveness of the weighted identification. Consider the parallel inverted pendulums system shown in Fig.6 The input of this system is the torque of the motor, and outputs are the rotation angles of the pendulum 1, 2 and the arm. This system has one input and three outputs. By considering the equation of motion, the transfer function is given as follows.

$$P_{m1} = \frac{1}{Den(s)}$$

$$\times \begin{bmatrix} -1.52s^2(s+6.66)(s-6.64) \\ -2.96s^2(s+4.78)(s-4.77) \\ 1.28(s+6.66)(s-6.64)(s+4.78)(s-4.77) \end{bmatrix} \quad (53)$$

$$Den(s) = s(s+1.62)(s+6.86)(s-6.75) \times (s+4.97)(s-4.87) \quad (54)$$

However, P_{m1} has model uncertainty. And we set the noises v and w, sampling time T, j and N as follows :

$$v,\ w \ :\ \text{white noise} \quad (55)$$

$$T = 0.005,\ j = 30,\ N = 8000 \quad (56)$$

Furthermore W_s and W_t in (4) is as follows.

$$W_s = \begin{bmatrix} W_{s1} & 0 & 0 \\ 0 & W_{s2} & 0 \\ 0 & 0 & W_{s3} \end{bmatrix} \quad (57)$$

$$W_{s1} = \frac{1000(s+20)(s+60)}{(s+0.1)(s+0.01)} \quad (58)$$

$$W_{s2} = \frac{1200(s+10)(s+50)}{(s+0.1)(s+0.01)} \quad (59)$$

$$W_{s3} = \frac{1300(s+80)}{s+1} \quad (60)$$

$$W_t = \frac{2(s+40)}{s+300} \quad (61)$$

For the comparison of the weighted and unweighted identification, we make use of the following two methods.

[Method 1] Joint design of weighted model subspace based state space identification and control.

[Method 2] Iterative design of *unweighted* model subspace based state space identification and control.

Fig.7 shows the value of J_c in (4) on five iterations. The minimum of J_c is equal to 2984 and solid line of Fig.7. Fig.8 and Fig.9 shows the gain plot of the second model P_{m2} identified by method 1 and method 2 respectively. Because W_s is a low-pass weighting function, the obtained model is accurate in the lower frequency domain by method 1. From this result, we can show the effectiveness of the weighted model subspace based state space identification.

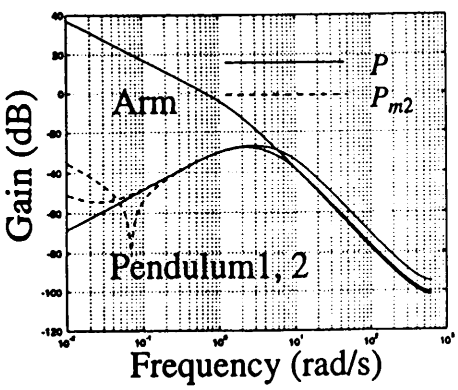

Figure 8: Gain plot of P_{m2} (method 1)

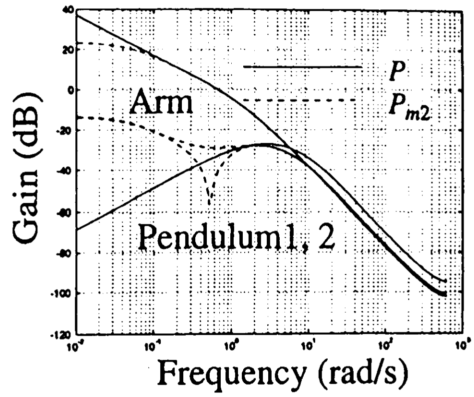

Figure 9: Gain plot of P_{m2} (method 2)

6 Conclusions

In this paper, we have proposed a ne state-space system identification method. The merits f this method are that the noise attenuation has been a ieved dispite the closed-loop identification, and that th cost function of this method can be regarded as the H_2 control spec. We have achieved these merits by using t e nominal model data as the instrumental variable an attenuating the noise from not only the output signal but also the input signal in the closed-loop identification. Furthermore, by using this method, we have proposed a joint design method of identification and control, and have shown the effectiveness of our method by numerical example.

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Appendix

A. The algorithm of model subspace based state space identification

[Step 1] Determine the $\Delta_1 \sim \Delta_4$ from

$$\Delta_{1\sim4} = \arg\min_{\Delta_{1\sim4}} \left\| \begin{bmatrix} Y_p \\ U_p \end{bmatrix} - \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{bmatrix} \begin{bmatrix} X_{m1} \\ U_{m1} \end{bmatrix} \right\| \tag{A.1}$$

then, since v and w are uncorrelation with r, if the nominal model P_{m1} and P are sufficiently close,

$$\lim_{N\to\infty} (\Delta_3 X_m + \Delta_4 U_m) = \widehat{U}_p^d \tag{A.2}$$

is satisfied.

[Step 2] Compute L and M from

$$L := \Delta_1 - \Delta_2\Delta_4^{-1}\Delta_3 \tag{A.3}$$
$$M := \Delta_2\Delta_4^{-1} \tag{A.4}$$

[Step 3] From (31) and (32), $\widehat{A}_p$ and $\widehat{C}_p$ are determined as follows.

$$\widehat{A}_p = L^{\dagger}(1:\ell(j-1))L(\ell+1:\ell j,:) \tag{A.5}$$

$$\widehat{C}_p = \sum_{k=1}^{j} L(\ell(k-1)+1:\ell k,:) \left[\sum_{k=1}^{j} \widehat{A}_p^{k-1}\right]^{\dagger} \tag{A.6}$$

[Step 4] From (33), we obtain

$$\Phi = \Theta \begin{bmatrix} I_\ell & 0 \\ 0 & L(1:m(j-1),:) \end{bmatrix} \begin{bmatrix} \widehat{D}_p \\ \widehat{B}_p \end{bmatrix} \tag{A.7}$$

where

$$\Phi := \begin{bmatrix} \Xi(:,1:m) \\ \Xi(:,m+1:2m) \\ \vdots \\ \Xi(:,m(j-1):mj) \end{bmatrix}, \quad \Xi := L^{\perp}M \tag{A.8}$$

$$\Theta := \begin{bmatrix} L^{\perp}(1:\ell,:)^T & \cdots & L^{\perp}(\ell(j-1)+1:\ell j,:))^T \\ L^{\perp}(\ell+1:2\ell,:)^T & \cdots & 0 \\ \vdots & & \vdots \\ L^{\perp}(\ell(j-1)+1:\ell j,:)^T & \cdots & 0 \end{bmatrix} \tag{A.9}$$

Then, $\widehat{D}_p$ and $\widehat{B}_p$ are computed as follows

$$\begin{bmatrix} \widehat{D}_p \\ \widehat{B}_p \end{bmatrix} = \left[\Theta \begin{bmatrix} I_\ell & 0 \\ 0 & L(1:m(j-1),:) \end{bmatrix}\right]^{\dagger} \Phi \tag{A.10}$$